Laplace transforms 2

Frames 1 to 42

Learning outcomes

When you have completed this Programme you will be able to:

- Use the Heaviside unit step function to 'switch' expressions on and off
- Obtain the Laplace transform of expressions involving the Heaviside unit step function

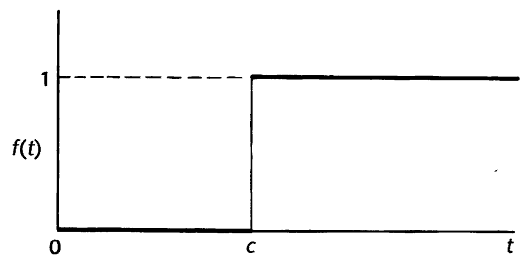
Introduction

1

In the previous Programme, we dealt with the Laplace transforms of continuous functions of t. In practical applications, it is convenient to have a function which, in effect, 'switches on' or 'switches off' a given term at pre-described values of t. This we can do with the *Heaviside unit step function.*

Heaviside unit step function

Consider a function that maintains a zero value for all values of t up to $t = c$ and a unit value for $t = c$ and all values of $t \geq c$.



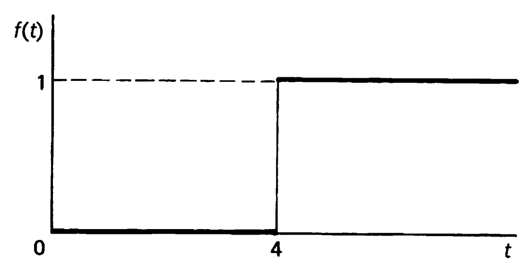
$f(t) = 0 \quad \text{for } t < c$
$f(t) = 1 \quad \text{for } t \geq c$

This function is the *Heaviside unit step function* and is denoted by

$f(t) = u(t - c)$

where the c indicates the value of t at which the function changes from a value of 0 to a value of 1.

Thus, the function



is denoted by $f(t) = \ldots\ldots\ldots\ldots$

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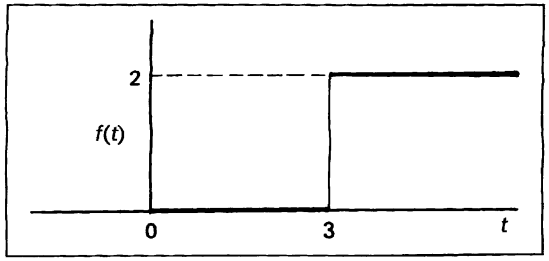
$$\boxed{f(t) = u(t - 4)}$$

Similarly, the graph of $f(t) = 2u(t - 3)$ is

$\ldots\ldots\ldots\ldots$

3



So $u(t-c)$ has just two values

for $t < c$, $u(t-c) = \dots\dots\dots\dots$
for $t \geq c$, $u(t-c) = \dots\dots\dots\dots$

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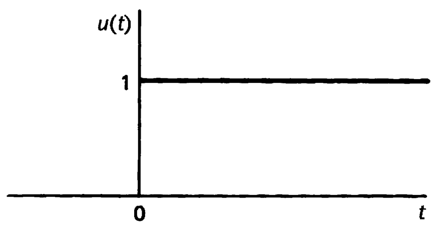
$$\boxed{t < c,\ u(t-c) = 0; \quad t \geq c,\ u(t-c) = 1}$$

Unit step at the origin

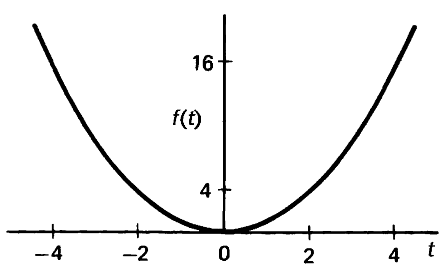


If the unit step occurs at the origin, then $c = 0$ and $f(t) = u(t-c)$ becomes

$f(t) = u(t)$

i.e. $u(t) = 0$ for $t < 0$
$u(t) = 1$ for $t \geq 0$.

Effect of the unit step function



The graph of $f(t) = t^2$ is, of course, as shown.

Remembering the definition of $u(t-c)$, the graph of

$f(t) = u(t-2) \cdot t^2$ is

$\dots\dots\dots\dots$

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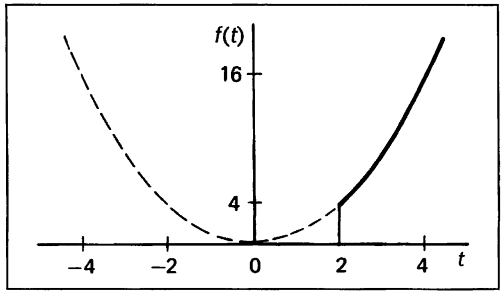

For $t < 2$, $u(t-2) = 0 \quad \therefore \; u(t-2) \cdot t^2 = 0 \cdot t^2 = 0$

$t \geq 2$, $u(t-2) = 1 \quad \therefore \; u(t-2) \cdot t^2 = 1 \cdot t^2 = t^2$

So the function $u(t-2)$ suppresses the function t^2 for all values of t up to $t = 2$ and 'switches on' the function t^2 at $t = 2$.

Now we can sketch the graphs of the following functions.

(a) $f(t) = \sin t \quad$ for $0 < t < 2\pi$

(b) $f(t) = u(t - \pi/4) \cdot \sin t \quad$ for $0 < t < 2\pi$.

These give ....

and ....

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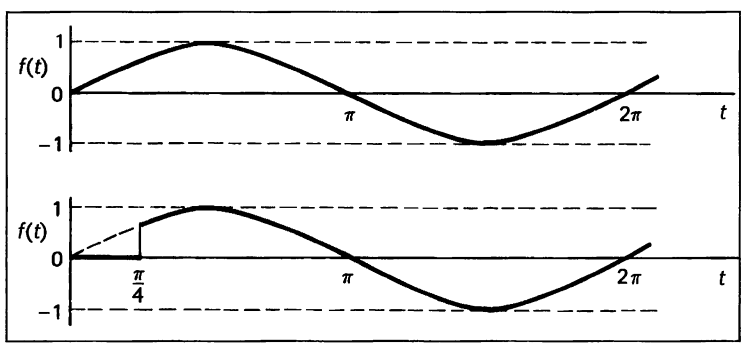
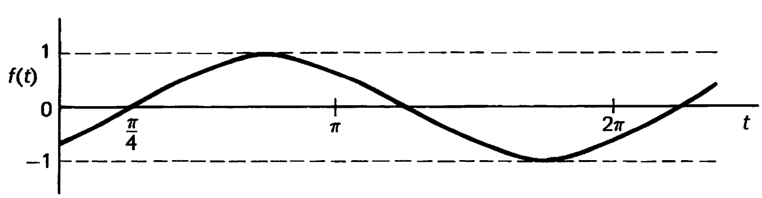


That is, the graph of $f(t) = u(t - \pi/4) \cdot \sin t$ is the graph of $f(t) = \sin t$ but suppressed for all values prior to $t = \pi/4$.

If we sketch the graph of $f(t) = \sin(t - \pi/4)$ we have



Since $u(t-c)$ has the effect of suppressing a function for $t < c$, then the graph of $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ is

....

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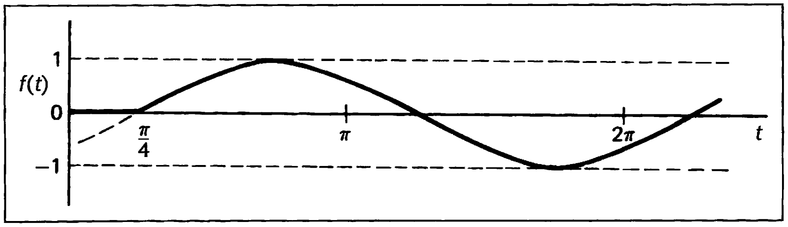

That is, the graph of $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ is the graph of $f(t) = \sin t$ $(t > 0)$, shifted $\pi/4$ units along the t-axis.

In general, the graph of $f(t) = u(t - c) \cdot \sin(t - c)$ is the graph of $f(t) = \sin t$ $(t > 0)$, shifted along the t-axis through an interval of c units.

Similarly, for $t > 0$, sketch the graphs of

(a) $f(t) = e^{-t}$

(b) $f(t) = u(t - c) \cdot e^{-t}$

(c) $f(t) = u(t - c) \cdot e^{-(t-c)}$

(d) $f(t) = e^{-t}\{u(t - 1) - u(t - 2)\}$.

Arrange the graphs under each other to show the important differences.

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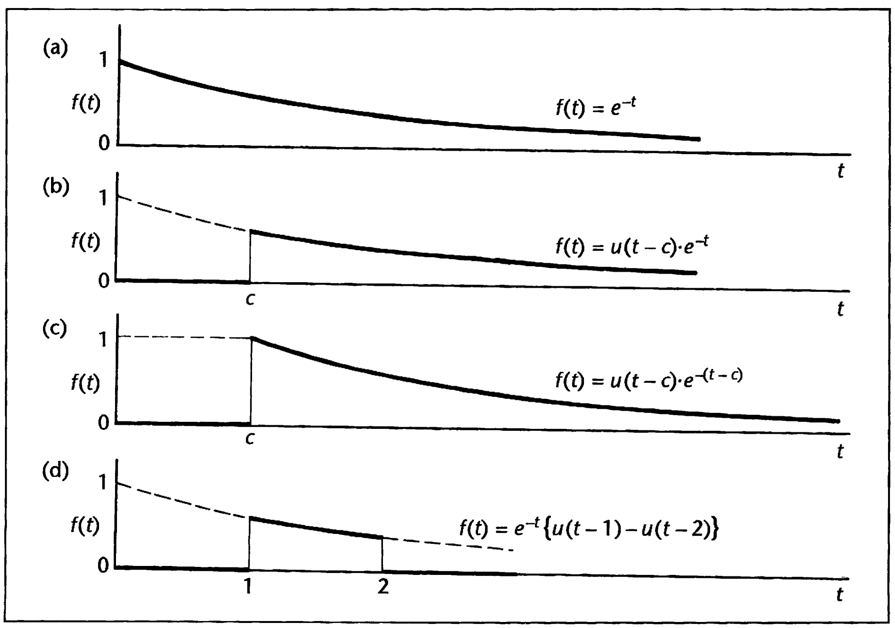

In (a), we have the graph of $f(t) = e^{-t}$

In (b), the same graph is suppressed prior to $t = c$

In (c), the graph of $f(t) = e^{-t}$ is shifted c units along the t-axis

In (d), the graph of $f(t) = e^{-t}$ is turned on at $t = 1$ and off at $t = 2$.

Laplace transform of $u(t-c)$

$$L\{u(t-c)\} = \frac{e^{-cs}}{s}$$

Because

$$L\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)\,\mathrm{d}t$$

but

$$e^{-st}u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)\,\mathrm{d}t = \int_c^\infty e^{-st}\,\mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_c^\infty = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of the unit step at the origin is

$$L\{u(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{1}{s}}$$

Because $c = 0$.

So $\quad L\{u(t-c)\} = \dfrac{e^{-cs}}{s}$

and $\quad L\{u(t)\} = \dfrac{1}{s}.$

Also from the definition of $u(t)$:

$$L(1) = L\{1 \cdot u(t)\}$$
$$L(t) = L\{t \cdot u(t)\}$$
$$L\{f(t)\} = L\{f(t) \cdot u(t)\}$$

Make a note of these results: we shall be using them

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As we have seen, the unit step function $u(t-c)$ is often combined with other functions of t, so we now consider the Laplace transform of $u(t-c)\cdot f(t-c)$.

Laplace transform of $u(t-c)\cdot f(t-c)$ (the second shift theorem)

$$L\{u(t-c)\cdot(f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s)$$

Because

$$L\{u(t-c)\cdot f(t-c)\} = \int_0^\infty e^{-st}u(t-c)\cdot f(t-c)\,\mathrm{d}t$$

$$\text{but } e^{-st}u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t-c)\cdot f(t-c)\} = \int_c^\infty e^{-st}f(t-c)\,\mathrm{d}t$$

We now make the substitution $t-c=v$ so that $t=c+v$ and $\mathrm{d}t=\mathrm{d}v$. Also for the limits, when $t=c$, $v=0$ and when $t\to\infty$, $v\to\infty$. Therefore

$$L\{u(t-c)\cdot f(t-c)\} = \int_0^\infty e^{-s(c+v)}f(v)\,\mathrm{d}v$$
$$= e^{-cs}\int_0^\infty e^{-sv}f(v)\,\mathrm{d}v$$

Now $\int_0^\infty e^{-sv}f(v)\,\mathrm{d}v$ has exactly the same value as $\int_0^\infty e^{-st}f(t)\,\mathrm{d}t$ which is, of course, the Laplace transform of $f(t)$. Therefore

$$L\{u(t-c)\cdot f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s)$$

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$$\boxed{L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s) \quad \text{where } F(s) = L\{f(t)\}}$$

$$\text{So } L\left\{u(t-4)\cdot(t-4)^2\right\} = e^{-4s}\cdot F(s) \quad \text{where } F(s) = L\{t^2\}$$
$$= e^{-4s}\left(\frac{2!}{s^3}\right) = \frac{2e^{-4s}}{s^3}$$

Note that $F(s)$ is the transform of t^2 and *not* of $(t-4)^2$.

In the same way:

$$L\{u(t-3)\cdot\sin(t-3)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{e^{-3s}}{s^2+1}}$$

Because $L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\cdot F(s)$ where $F(s) = L\{\sin t\}$

$$= \frac{1}{s^2+1}$$

$$\therefore\ L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\left(\frac{1}{s^2+1}\right)$$

So now do these in the same way.

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = \ldots\ldots\ldots\ldots$

(b) $L\{u(t-1)\cdot\sin 3(t-1)\} = \ldots\ldots\ldots\ldots$

(c) $L\left\{u(t-5)\cdot e^{(t-5)}\right\} = \ldots\ldots\ldots\ldots$

(d) $L\{u(t-\pi/2)\cdot\cos 2(t-\pi/2)\} = \ldots\ldots\ldots\ldots$

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Here they are

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = e^{-2s}\cdot F(s)$ where $F(s) = L\{t^3\}$

$$= e^{-2s}\left(\frac{3!}{s^4}\right) = \frac{6e^{-2s}}{s^4}$$

(b) $L\{u(t-1)\cdot\sin 3(t-1)\} = e^{-s}\cdot F(s)$ where $F(s) = L\{\sin 3t\}$

$$= e^{-s}\left(\frac{3}{s^2+9}\right) = \frac{3e^{-s}}{s^2+9}$$

(c) $L\left\{u(t-5)\cdot e^{(t-5)}\right\} = e^{-5s}\cdot F(s)$ where $F(s) = L\{e^t\}$

$$= e^{-5s}\left(\frac{1}{s-1}\right) = \frac{e^{-5s}}{s-1}$$

(d) $L\{u(t-\pi/2)\cdot\cos 2(t-\pi/2)\} = e^{-\pi s/2}\cdot F(s)$ where $F(s) = L\{\cos 2t\}$

$$= e^{-\pi s/2}\left(\frac{s}{s^2+4}\right) = \frac{s\cdot e^{-\pi s/2}}{s^2+4}$$

So $L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s)$ where $F(s) = L\{f(t)\}$.
Written in reverse, this becomes

If $F(s) = L\{f(t)\}$, then $e^{-cs}\cdot F(s) = L\{u(t-c)\cdot f(t-c)\}$

where c is real and positive.

This is known as the *second shift theorem.*

Make a note of it: then we will use it

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$$\boxed{\text{If } F(s) = L\{f(t)\}, \text{ then } e^{-cs} \cdot F(s) = L\{u(t-c) \cdot f(t-c)\}}$$

This is useful in finding inverse transforms, as we shall now see.

Example 1

Find the function whose transform is $\dfrac{e^{-4s}}{s^2}$.

The numerator corresponds to e^{-cs} where $c = 4$ and therefore indicates $u(t-4)$.

Then $\dfrac{1}{s^2} = F(s) = L\{t\} \quad \therefore \; f(t) = t$.

$$\therefore \; L^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} = u(t-4) \cdot (t-4)$$

Remember that in writing the final result, $f(t)$ is replaced by

....

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$$\boxed{f(t-c)}$$

Example 2

Determine $L^{-1}\left\{\dfrac{6e^{-2s}}{s^2+4}\right\}$.

The numerator contains e^{-2s} and therefore indicates ....

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$$\boxed{u(t-2)}$$

The remainder of the transform, i.e. $\dfrac{6}{s^2+4}$, can be written as $3\left(\dfrac{2}{s^2+4}\right)$

$$\therefore \; \frac{6}{s^2+4} = F(s) = L\{\ldots\ldots\ldots\ldots\}$$

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$$\boxed{L\{3\sin 2t\}}$$

$$\therefore \; L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{3u(t-2)\cdot\sin 2(t-2)}$$

Because

$$L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\} = u(t-2)\cdot f(t-2) \quad \text{where } f(t) = L^{-1}\left\{\frac{6}{s^2+4}\right\}$$
$$= u(t-2)\cdot 3\sin 2(t-2)$$

Example 3

Determine $L^{-1}\left\{\dfrac{s\cdot e^{-s}}{s^2+9}\right\}$.

This, in similar manner, is ....

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$$\boxed{u(t-1)\cdot\cos 3(t-1)}$$

Because the numerator contains e^{-s} which indicates $u(t-1)$.

Also $\dfrac{s}{s^2+9} = F(s) = L\{\cos 3t\}$

$\therefore\ f(t) = \cos 3t \quad \therefore\ f(t-1) = \cos 3(t-1)$.

$$\therefore\ L^{-1}\left\{\frac{s\cdot e^{-s}}{s^2+9}\right\} = u(t-1)\cdot\cos 3(t-1)$$

Remember that, having obtained $f(t)$, the result contains $f(t-c)$.

Here is a short exercise by way of practice.

Exercise

Determine the inverse transforms of the following.

(a) $\dfrac{2e^{-5s}}{s^3}$ (d) $\dfrac{2s\cdot e^{-3s}}{s^2-16}$

(b) $\dfrac{3e^{-2s}}{s^2-1}$ (e) $\dfrac{5e^{-s}}{s}$

(c) $\dfrac{8e^{-4s}}{s^2+4}$ (f) $\dfrac{s\cdot e^{-s/2}}{s^2+2}$

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Results – all very straightforward.

(a) $u(t-5)\cdot(t-5)^2$

(b) $3u(t-2)\cdot\sinh(t-2)$

(c) $4u(t-4)\cdot\sin 2(t-4)$

(d) $2u(t-3)\cdot\cosh 4(t-3)$

(e) $5u(t-1)$

(f) $u(t-1/2)\cdot\cos\sqrt{2}(t-1/2)$.

Before looking at a more interesting example, let us collect our results together as far as we have gone.

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The main points are

(a) $u(t-c) = 0 \quad 0 < t < c$
$\quad\;\; = 1 \quad t \geq c$ (1)

(b) $L\{u(t-c)\} = \dfrac{e^{-cs}}{s}$
$\quad L\{u(t)\} = \dfrac{1}{s}$ (2)

(c) $L\{u(t-c) \cdot f(t-c)\} = e^{-cs} \cdot F(s)$ where $F(s) = L\{f(t)\}$ (3)

(d) If $F(s) = L\{f(t)\}$, then $e^{-cs} \cdot F(s) = L\{u(t-c)\} \cdot f(t-c)\}$ (4)

Now let us apply these to some further examples.

Example 1

Determine the expression $f(t)$ for which

$$L\{f(t)\} = \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}$$

We take each term in turn and find its inverse transform.

(a) $L^{-1}\left\{\dfrac{3}{s}\right\} = 3L^{-1}\left\{\dfrac{1}{s}\right\} = 3$ i.e. $3u(t)$

(b) $L^{-1}\left\{\dfrac{4e^{-s}}{s^2}\right\} = u(t-1) \cdot 4(t-1)$

(c) $L^{-1}\left\{\dfrac{5e^{-2s}}{s^2}\right\} = \dots\dots\dots\dots$

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$$\boxed{u(t-2) \cdot 5(t-2)}$$

So we have $L^{-1}\left\{\dfrac{3}{s}\right\} = 3u(t)$

$$L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1) \cdot 4(t-1)$$

$$L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = u(t-2) \cdot 5(t-2)$$

$$\therefore\; F(t) = 3u(t) - u(t-1) \cdot 4(t-1) + u(t-2) \cdot 5(t-2)$$

To sketch the graph of $f(t)$ we consider the values of the function within the three sections $0 < t < 1$, $1 < t < 2$, and $2 < t$.

Between $t = 0$ and $t = 1$, $f(t) = \dots\dots\dots\dots$

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$$\boxed{f(t) = 3}$$

Because in this interval, $u(t) = 1$, but $u(t-1) = 0$ and $u(t-2) = 0$. In the same way, between $t = 1$ and $t = 2$, $f(t) = \dots\dots\dots\dots$

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$$\boxed{f(t) = 7 - 4t}$$

Because between $t = 1$ and $t = 2$, $u(t) = 1$, $u(t-1) = 1$, but $u(t-2) = 0$.

$\therefore\ f(t) = 3 - 4(t-1) + 0 = 3 - 4t + 4 = 7 - 4t$

Similarly, for $t > 2$, $f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{f(t) = t - 3}$$

Because for $t > 2$, $u(t) = 1$, $u(t-1) = 1$ and $u(t-2) = 1$

$$\begin{aligned} \therefore\ f(t) &= 3 - 4(t-1) + 5(t-2) \\ &= 3 - 4t + 4 + 5t - 10 = t - 3 \end{aligned}$$

So, collecting the results together, we have

for $0 < t < 1$, $f(t) = 3$
$1 < t < 2$, $f(t) = 7 - 4t$ $(t = 1, f(t) = 3;\ t = 2, f(t) = -1)$
$2 < t$, $f(t) = t - 3$ $(t = 2, f(t) = -1;\ t = 3, f(t) = 0)$

Using these facts we can sketch the graph of $f(t)$, which is

$\ldots\ldots\ldots\ldots$

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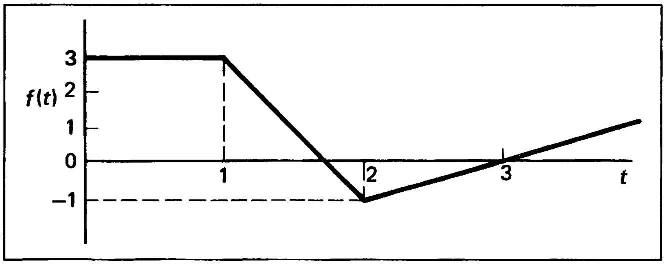

Here is another.

Example 2

Determine the expression $f(t) = L^{-1}\left\{\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right\}$ and sketch the graph of $f(t)$.

First we express the inverse transform of each term in terms of the unit step function.

This gives $\ldots\ldots\ldots\ldots$

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$$L^{-1}\left\{\frac{2}{s}\right\} = 2u(t); \quad L^{-1}\left\{\frac{3e^{-s}}{s^2}\right\} = u(t-1)\cdot 3(t-1)$$
$$L^{-1}\left\{\frac{3e^{-3s}}{s^2}\right\} = u(t-3)\cdot 3(t-3)$$

$$\therefore \ f(t) = 2u(t) + u(t-1)\cdot 3(t-1) - u(t-3)\cdot 3(t-3)$$

So there are 'break points', i.e. changes of function, at $t = 1$ and $t = 3$, and we investigate $f(t)$ within the three intervals.

$0 < t < 1 \quad f(t) = \ldots\ldots\ldots\ldots$
$1 < t < 3 \quad f(t) = \ldots\ldots\ldots\ldots$
$3 < t \quad f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{0 < t < 1,\ f(t) = 2; \quad 1 < t < 3,\ f(t) = 3t - 1; \quad 3 < t,\ f(t) = 8}$$

Because with

$0 < t < 1, \quad u(t) = 1$, but $u(t-1) = u(t-3) = 0 \qquad \therefore\ f(t) = 2$
$1 < t < 3, \quad u(t) = 1,\ u(t-1) = 1$, but $u(t-3) = 0$
$\qquad \therefore\ f(t) = 2 + 3(t-1) = 3t - 1 \qquad \therefore\ f(t) = 3t - 1$
$3 < t, \quad u(t) = 1,\ u(t-1) = 1,\ u(t-3) = 1$
$\qquad \therefore\ f(t) = 2 + 3t - 3 - 3t + 9 \qquad \therefore\ f(t) = 8$

Therefore, the graph of $f(t)$ is ....

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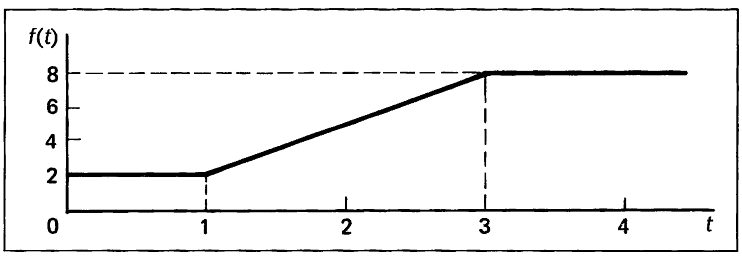

Between the break points, $f(t) = 3t - 1 \quad \begin{cases} t = 1, f(t) = 2 \\ t = 3, f(t) = 8 \end{cases}$

Now move on for the next example

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Example 3

If $f(t) = L^{-1}\left\{\dfrac{(1 - e^{-2s})(1 + e^{-4s})}{s^2}\right\}$, determine $f(t)$ and sketch the graph of the function.

Although at first sight this looks more complicated, we simply multiply out the numerator and proceed as before.

$$f(t) = L^{-1}\left\{\frac{1 - e^{-2s} + e^{-4s} - e^{-6s}}{s^2}\right\}$$
$$= L^{-1}\left\{\frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2} - \frac{e^{-6s}}{s^2}\right\}$$

We now write down the inverse transform of each term in terms of the unit function, so that

$$f(t) = \ldots\ldots\ldots\ldots$$

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$$\boxed{f(t) = u(t)\cdot t - u(t-2)\cdot(t-2) + u(t-4)\cdot(t-4) - u(t-6)\cdot(t-6)}$$

and we can see there are break points at $t = 2$, $t = 4$, $t = 6$.

For $0 < t < 2$, $f(t) = t - 0 + 0 - 0$ $\quad f(t) = t$

$2 < t < 4$, $f(t) = t - (t-2) + 0 - 0$ $\quad f(t) = 2$

$4 < t < 6$, $f(t) = t - (t-2) + (t-4) - 0$ $\quad f(t) = t - 2$

$6 < t$, $f(t) = t - (t-2) + (t-4) - (t-6)$ $\quad f(t) = 4$

The second and fourth components are constant, but before sketching the graph of the function, we check the values of $f(t) = t$ and $f(t) = t - 2$ at the relevant break points.

$f(t) = t$. At $t = 0$, $f(t) = 0$; at $t = 2$, $f(t) = 2$

$f(t) = t - 2$. At $t = 4$, $f(t) = 2$; at $t = 6$, $f(t) = 4$.

So the graph of the function is ....

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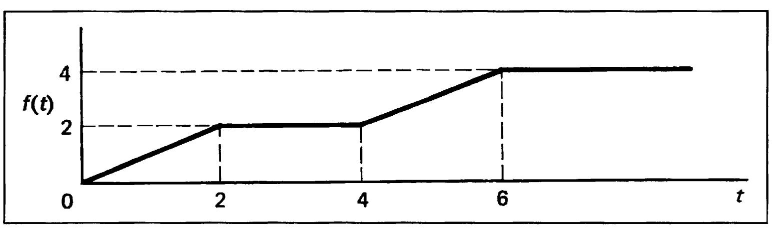

It is always wise to calculate the function values at break points, since discontinuities, or jumps, sometimes occur.

On to the next frame

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Now for one in reverse.

Example 4

A function $f(t)$ is defined by

$f(t) = 4$ for $0 < t < 2$

$= 2t - 3$ for $2 < t$.

Sketch the graph of the function and determine its Laplace transform.

We see that for $t = 0$ to $t = 2$, $f(t) = 4$.

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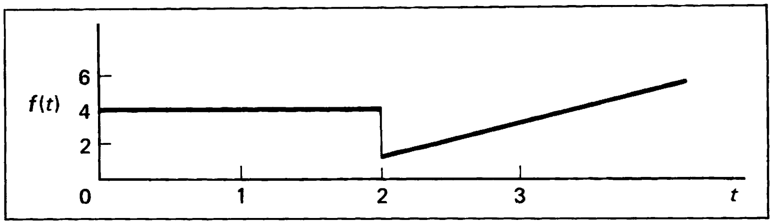

Notice the discontinuity at $t = 2$.

Expressing the function in unit step form:

$$f(t) = 4u(t) - 4u(t-2) + u(t-2) \cdot (2t-3)$$

Note that the second term cancels $f(t) = 4$ at $t = 2$ and that the third switches on $f(t) = 2t - 3$ at $t = 2$.

Before we can express this in Laplace transforms, $(2t-3)$ in the third term must be written as a function of $(t-2)$ to correspond to $u(t-2)$. Therefore, we write $2t - 3$ as $2(t-2) + 1$.

$$\begin{aligned}\text{Then } f(t) &= 4u(t) - 4u(t-2) + u(t-2) \cdot \{2(t-2) + 1\} \\ &= 4u(t) - 4u(t-2) + u(t-2) \cdot 2(t-2) + u(t-2) \\ &= 4u(t) - 3u(t-2) + u(t-2) \cdot 2(t-2)\end{aligned}$$

$$\therefore\ L\{f(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{f(t)\} = \frac{4}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-2s}}{s^2}}$$

Here is one for you to work through in much the same way.

Example 5

A function is defined by

$$\begin{aligned} f(t) &= 6 \quad & 0 < t < 1 \\ &= 8 - 2t \quad & 1 < t < 3 \\ &= 4 \quad & 3 < t. \end{aligned}$$

Sketch the graph and find the Laplace transform of the function.

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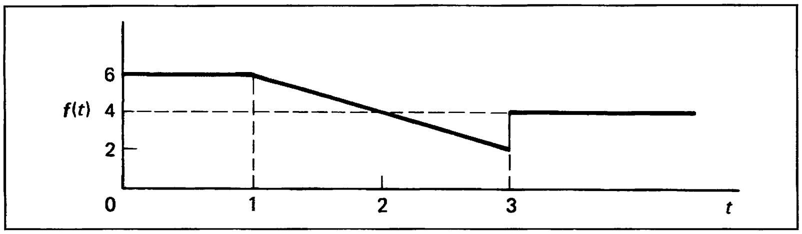

Expressing this in unit step form we have

$$f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot(8-2t)$$
$$- u(t-3)\cdot(8-2t) + u(t-3)\cdot 4$$

where the second term switches off the first function $f(t) = 6$ at $t = 1$ and the third term switches on the second function $f(t) = 8 - 2t$, which in turn is switched off by the fourth term at $t = 3$ and replaced by $f(t) = 4$ in the fifth term.

Before we can write down the transforms of the third and fourth terms, we must express $f(t) = 8 - 2t$ in terms of $(t-1)$ and $(t-3)$ respectively.

$$8 - 2t = 6 + 2 - 2t = 6 - 2(t-1)$$
$$8 - 2t = 2 + 6 - 2t = 2 - 2(t-3)$$
$$\therefore\ f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot\{6 - 2(t-1)\}$$
$$- u(t-3)\cdot\{2 - 2(t-3)\} + 4u(t-3)$$
$$= 6u(t) - 6u(t-1) + 6u(t-1)$$
$$- u(t-1)\cdot 2(t-1) - 2u(t-3)$$
$$+ u(t-3)\cdot 2(t-3) + 4u(t-3)$$

which simplifies finally to $f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{f(t) = 6u(t) - u(t-1)\cdot 2(t-1) + u(t-3)\cdot 2(t-3) + 2u(t-3)}$$

from which $L\{f(t)\} = \ldots\ldots\ldots\ldots$

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$$\boxed{L\{f(t)\} = \frac{6}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} + \frac{2e^{-3s}}{s}}$$

Note that, in building up the function in unit step form

(a) to 'switch on' a function $f(t)$ at $t = c$, we add the term $u(t-c)\cdot f(t-c)$

(b) to 'switch off' a function $f(t)$ at $t = c$, we subtract $u(t-c)\cdot f(t-c)$.

You have now reached the end of this Programme and this brings you to the **Revision summary** and the **Can You?** checklist. Following that is the **Test exercise**. Work through this *at your own pace*. A set of **Further problems** provides additional valuable practice.

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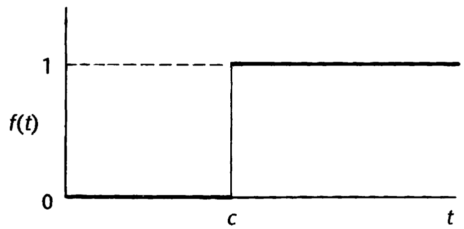
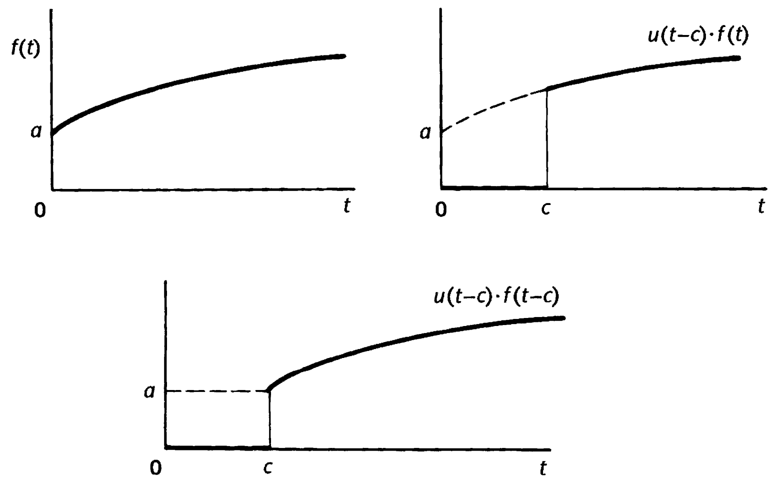
Revision summary 3

1 *Heaviside unit step function: $u(t-c)$*



$$f(t) = 0 \qquad 0 < t < c$$
$$\quad = 1 \qquad c < t$$

2 *Suppression and shift*



3 *Laplace transform of $u(t-c)$*

$$L\{u(t-c)\} = \frac{e^{-cs}}{s}; \quad L\{u(t)\} = \frac{1}{s}.$$

4 *Laplace transform of $u(t-c)\cdot f(t-c)$*

$$L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s) \quad \text{where } F(s) = L\{f(t)\}.$$

5 *Second shift theorem*

If $F(s) = L\{f(t)\}$, then $e^{-cs}\cdot F(s) = L\{u(t-c)\cdot f(t-c)\}$ where c is real and positive.

Can You?

Checklist 3

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident you are that you can: Frames

- Use the Heaviside unit step function to 'switch' expressions on and off? 1 to 8

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the Laplace transform of expressions involving the Heaviside unit step function? 8 to 38

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 3

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1 In each of the following cases, sketch the graph of the function and find its Laplace transform.

(a) $f(t) = 3t \quad 0 < t < 2$
$ = 6 \quad 2 < t$

(b) $f(t) = e^{-2t} \quad 0 < t < 3$
$ = 0 \quad 3 < t$

(c) $f(t) = t^2 \quad 0 < t < 2$
$ = 2 \quad 2 < t < 3$
$ = 4 \quad 3 < t$

(d) $f(t) = \sin 2t \quad 0 < t < \pi$
$ = 0 \quad \pi < t.$

2 Determine the function $f(t)$ whose transform $F(s)$ is

$$F(s) = \frac{1}{s}\left\{2 - 5e^{-s} + 8e^{-3s}\right\}.$$

Sketch the graph of the function between $t = 0$ and $t = 4$.

3 If $f(t) = L^{-1}\left\{\dfrac{(1+3e^{-2s})(1-e^{-3s})}{s^2}\right\}$, determine $f(t)$ and sketch the graph of the function.

4 Determine the function $f(t)$ for which

$$f(t) = L^{-1}\left\{\frac{2(1-e^{-s})}{s(1-e^{-3s})}\right\}.$$

Sketch the waveform and express the function in analytical form.

Further problems 3

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1 If $L\{f(t)\} = \dfrac{1}{s^2}\left\{3s + 2e^{-2s} - 2e^{-5s}\right\}$, determine $f(t)$.

2 If $f(t) = L^{-1}\left\{\dfrac{(1-e^{-s})(1+e^{-2s})}{s^2}\right\}$, find $f(t)$ in terms of the unit step function.

3 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= 4 \qquad && 0 < t < 3 \\ &= 2t+1 && 3 < t. \end{aligned}$$

Sketch the graph of the function and determine its Laplace transform.

4 Express in terms of the Heaviside unit step function

(a) $$\begin{aligned} f(t) &= t^2 \qquad && 0 < t < 3 \\ &= 5t && 3 < t. \end{aligned}$$

(b) $$\begin{aligned} f(t) &= \cos t \qquad && 0 < t < \pi \\ &= \cos 2t && \pi < t < 2\pi \\ &= \cos 3t && 2\pi < t. \end{aligned}$$

5 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= 0 \qquad && 0 < t < 2 \\ &= t+1 && 2 < t < 3 \\ &= 0 && 3 < t. \end{aligned}$$

Determine $L\{f(t)\}$.

6 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= t^2 \qquad && 0 < t < 2 \\ &= 4 && 2 < t < 5 \\ &= 0 && 5 < t. \end{aligned}$$

Determine (a) the function in terms of the unit step function

(b) the Laplace transform of $f(t)$.